CHAPTER 9

Multiple Integrals

Much of the procedure for double and triple integrals may be thought of as a reversal of partial differentiation and otherwise is analogous to that for single integrals. However, one complexity that must be addressed relates to the domain of definition. With single integrals, the functions of one variable were defined on intervals of real numbers. Thus, the integrals only depended on the properties of the functions. The integrands of double and triple integrals are functions of two and three variables, respectively, and as such are defined on two- and three-dimensional regions. These regions have a flexibility in shape not possible in the single-variable cases. For example, with functions of two variables, and the corresponding double integrals, rectangular regions, $a \leqq x \leqq b$, $c \leqq y \leqq d$ are common. However, in many problems the domains are regions bound above and below by segments of plane curves. In the case of functions of three variables, and the corresponding triple integrals other than the regions $a \leqq x \leqq b, c \leqq y \leqq d, e \leqq z \leqq f$, there are those bound above and below by portions of surfaces. In very special cases, double and triple integrals can be directly evaluated. However, the systematic technique of *iterated integration* is the usual procedure. It is here that the reversal of partial differentiation comes into play.

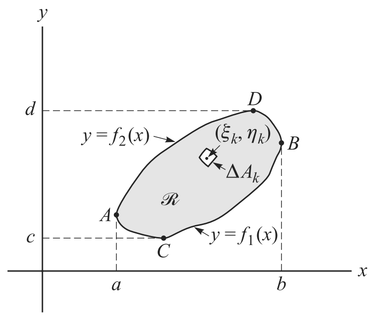

Fig. 9-1

Definitions of double and triple integrals are given below. Also, the method of iterated integration is described.

DOUBLE INTEGRALS

Let $F(x, y)$ be defined in a closed region $\mathscr{R}$ of the xy plane (see Fig. 9-1). Subdivide $\mathscr{R}$ into n subregions $\Delta\mathscr{R}_k$ of area ΔA_k, $k = 1, 2, \ldots, n$. Let (ξ_k, η_k) be some point of ΔA_k. Form the sum

$$\sum_{k=1}^{n} F(\xi_k, \eta_k)\,\Delta A_k \tag{1}$$

Consider

$$\lim_{n\to\infty} \sum_{k=1}^{n} F(\xi_k, \eta_k)\,\Delta A_k \tag{2}$$

where the limit is taken so that the number n of subdivisions increases without limit and such that the largest linear dimension of each ΔA_k approaches zero. See Fig. 9-2(a). If this limit exists, it is denoted by

$$\int_{\mathscr{R}}\int F(x, y)\, dA \tag{3}$$

and is called the *double integral* of $F(x, y)$ over the region $\mathscr{R}$.

It can be proved that the limit does exist if $F(x, y)$ is continuous (or sectionally continuous) in $\mathscr{R}$.

The double integral has a great variety of interpretations with any individual one dependent on the form of the integrand. For example, if $F(x, y) = \rho(x, y)$ represents the variable density of a flat iron plate then the double integral, $\int_A \rho\, dA$, of this function over a same shaped plane region, A, is the mass of the plate. In Fig. 9-2(b) we assume that $F(x, y)$ is a height function (established by a portion of a surface $z = F(x, y)$) for a cylindrically shaped object. In this case the double integral represents a volume.

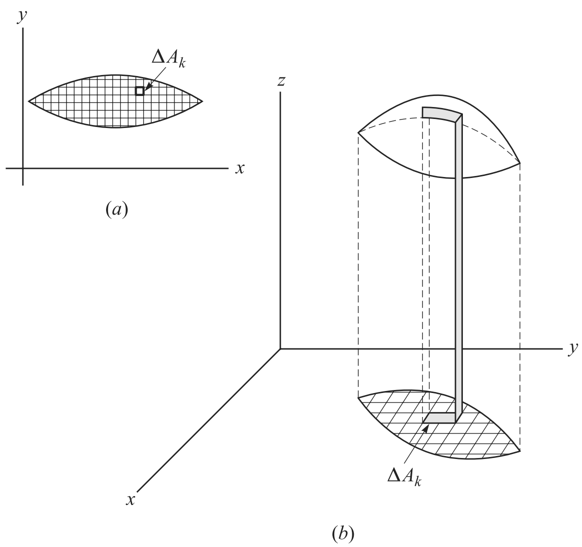

Fig. 9-2

ITERATED INTEGRALS

If $\mathscr{R}$ is such that any lines parallel to the y-axis meet the boundary of $\mathscr{R}$ in at most two points (as is true in Fig. 9-1), then we can write the equations of the curves ACB and ADB bounding $\mathscr{R}$ as $y = f_1(x)$ and $y = f_2(x)$, respectively, where $f_1(x)$ and $f_2(x)$ are single-valued and continuous in $a \leqq x \leqq b$. In this case we can evaluate the double integral (3) by choosing the regions $\Delta\mathscr{R}_k$ as rectangles formed by constructing a grid of lines parallel to the x- and y-axes and ΔA_k as the corresponding areas. Then (3) can be written

$$\iint_{\mathscr{R}} F(x, y)\, dx\, dy = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} F(x, y)\, dy\, dx \tag{4}$$

$$= \int_{x=a}^{b} \left\{ \int_{y=f_1(x)}^{f_2(x)} F(x, y)\, dy \right\} dx$$

where the integral in braces is to be evaluated first (keeping x constant) and finally integrating with respect to x from a to b. The result (*4*) indicates how a double integral can be evaluated by expressing it in terms of two single integrals called *iterated integrals*.

The process of iterated integration is visually illustrated in Fig. 9-3*a*,*b* and further illustrated as follows.

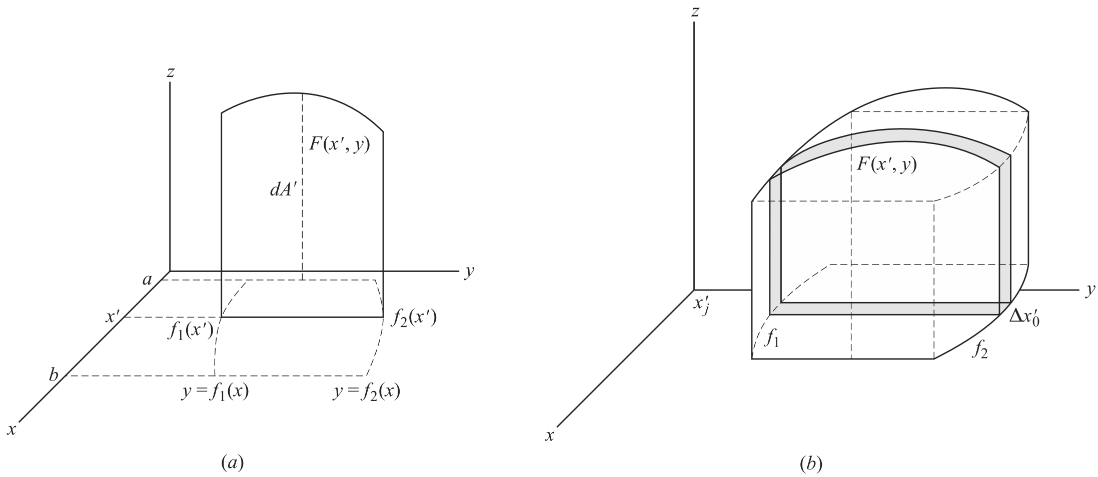

Fig. 9-3

The general idea, as demonstrated with respect to a given three-space region, is to establish a plane section, integrate to determine its area, and then add up all the plane sections through an integration with respect to the remaining variable. For example, choose a value of x (say, $x = x'$). The intersection of the plane $x = x'$ with the solid establishes the plane section. In it $z = F(x', y)$ is the height function, and if $y = f_1(x)$ and $y = f_2(x)$ (for all z) are the bounding cylindrical surfaces of the solid, then the width is $f_2(x') - f_1(x')$, i.e., $y_2 - y_1$. Thus, the area of the section is $A = \int_{y_1}^{y_2} F(x', y)\, dy$. Now establish slabs $A_j \Delta x_j$, where for each interval $\Delta x_j = x_j - x_{j-1}$, there is an intermediate value x_j'. Then sum these to get an approximation to the target volume. Adding the slabs and taking the limit yields

$$V = \lim_{n \to \infty} \sum_{j=1}^{n} A_j\, \Delta x_j = \int_a^b \left(\int_{y_1}^{y_2} F(x, y)\, dy \right) dx$$

In some cases the order of integration is dictated by the geometry. For example, if $\mathscr{R}$ is such that any lines parallel to the x-axis meet the boundary of $\mathscr{R}$ in at most two points (as in Fig. 9-1), then the equations of curves CAD and CBD can be written $x = g_1(y)$ and $x = g_2(y)$ respectively and we find similarly

$$\iint_{\mathscr{R}} F(x, y)\, dx\, dy = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} F(x, y)\, dx\, dy \tag{5}$$

$$= \int_{y=c}^{d} \left\{ \int_{x=g_1(y)}^{g_2(y)} F(x, y)\, dx \right\} dy$$

If the double integral exists, (*4*) and (*5*) yield the same value. (See, however, Problem 9.21.) In writing a double integral, either of the forms (*4*) or (*5*), whichever is appropriate, may be used. We call one form an *interchange of the order of integration* with respect to the other form.

In case $\mathscr{R}$ is not of the type shown in the above figure, it can generally be subdivided into regions $\mathscr{R}_1, \mathscr{R}_2, \ldots$ which are of this type. Then the double integral over $\mathscr{R}$ is found by taking the sum of the double integrals over $\mathscr{R}_1, \mathscr{R}_2, \ldots$.

TRIPLE INTEGRALS

The above results are easily generalized to closed regions in three dimensions. For example, consider a function $F(x, y, z)$ defined in a closed three-dimensional region $\mathscr{R}$. Subdivide the region into n subregions of volume ΔV_k, $k = 1, 2, \ldots, n$. Letting (ξ_k, η_k, ζ_k) be some point in each subregion, we form

$$\lim_{n \to \infty} \sum_{k=1}^{n} F(\xi_k, \eta_k, \zeta_k)\, \Delta V_k \tag{6}$$

where the number n of subdivisions approaches infinity in such a way that the largest linear dimension of each subregion approaches zero. If this limit exists, we denote it by

$$\iiint_{\mathscr{R}} F(x, y, z)\, dV \tag{7}$$

called the *triple integral* of $F(x, y, z)$ over $\mathscr{R}$. The limit does exist if $F(, x, y, z)$ is continuous (or piecemeal continuous) in $\mathscr{R}$.

If we construct a grid consisting of planes parallel to the xy, yz, and xz planes, the region $\mathscr{R}$ is subdivided into subregions which are rectangular parallelepipeds. In such case we can express the triple integral over $\mathscr{R}$ given by (7) as an *iterated integral* of the form

$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(a)} \int_{z=f_1(x,y)}^{f_2(x,y)} F(x, y, z)\, dx\, dy\, dz = \int_{x=a}^{b} \left[\int_{y=g_1(x)}^{g_2(x)} \left\{ \int_{z=f_1(x,y)}^{f_2(x,y)} F(x, y, z)\, dz \right\} dy \right] dx \tag{8}$$

(where the innermost integral is to be evaluated first) or the sum of such integrals. The integration can also be performed in any other order to give an equivalent result.

The iterated triple integral is a sequence of integrations; first from surface portion to surface portion, then from curve segment to curve segment, and finally from point to point. (See Fig. 9-4.)

Extensions to higher dimensions are also possible.

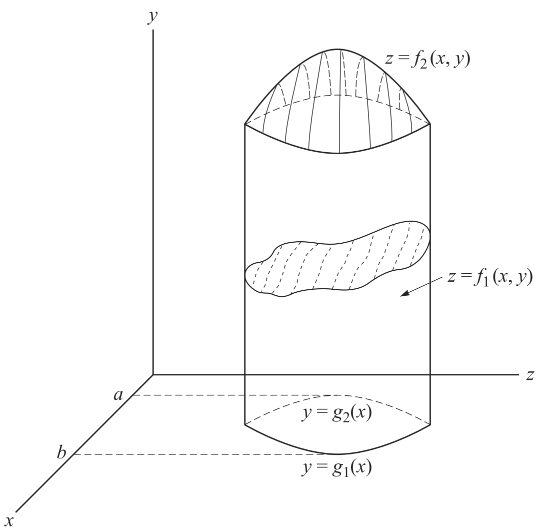

Fig. 9-4

TRANSFORMATIONS OF MULTIPLE INTEGRALS

In evaluating a multiple integral over a region $\mathscr{R}$, it is often convenient to use coordinates other than rectangular, such as the curvilinear coordinates considered in Chapters 6 and 7.

If we let (u, v) be curvilinear coordinates of points in a plane, there will be a set of transformation equations $x = f(u, v)$, $y = g(u, v)$ mapping points (x, y) of the xy plane into points (u, v) of the uv plane. In such case the region $\mathscr{R}$ of the xy plane is mapped into a region $\mathscr{R}'$ of the uv plane. We then have

$$\iint_{\mathscr{R}} F(x, y)\, dx\, dy = \iint_{\mathscr{R}'} G(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv \tag{9}$$

where $G(u, v) \equiv F\{f(u, v), g(u, v)\}$ and

$$\frac{\partial(x, y)}{\partial(u, v)} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \tag{10}$$

is the *Jacobian* of x and y with respect to u and v (see Chapter 6).

Similarly if (u, v, w) are curvilinear coordinates in three dimensions, there will be a set of transformation equations $x = f(u, v, w)$, $y = g(u, v, w)$, $z = h(u, v, w)$ and we can write

$$\iiint_{\mathscr{R}} F(x, y, z)\, dx\, dy\, dz = \iiint_{\mathscr{R}'} G(u, v, w) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw \tag{11}$$

where $G(u, v, w) \equiv F\{f(u, v, w), g(u, v, w), h(u, v, w)\}$ and

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \tag{12}$$

is the Jacobian of x, y, and z with respect to u, v, and w.

The results (*9*) and (*11*) correspond to change of variables for double and triple integrals.

Generalizations to higher dimensions are easily made.

THE DIFFERENTIAL ELEMENT OF AREA IN POLAR COORDINATES, DIFFERENTIAL ELEMENTS OF AREA IN CYLINDRAL AND SPHERICAL COORDINATES

Of special interest is the differential element of area, dA, for polar coordinates in the plane, and the differential elements of volume, dV, for cylindrical and spherical coordinates in three space. With these in hand the double and triple integrals as expressed in these systems are seen to take the following forms. (See Fig. 9-5.)

The transformation equations relating cylindrical coordinates to rectangular Cartesian ones appeared in Chapter 7, in particular,

$$x = \rho \cos \phi,\ y = \rho \sin \phi,\ z = z$$

The coordinate surfaces are circular cylinders, planes, and planes. (See Fig. 9-5.)

At any point of the space (other than the origin), the set of vectors $\left\{ \dfrac{\partial \mathbf{r}}{\partial \rho}, \dfrac{\partial \mathbf{r}}{\partial \phi}, \dfrac{\partial \mathbf{r}}{\partial z} \right\}$ constitutes an orthogonal basis.

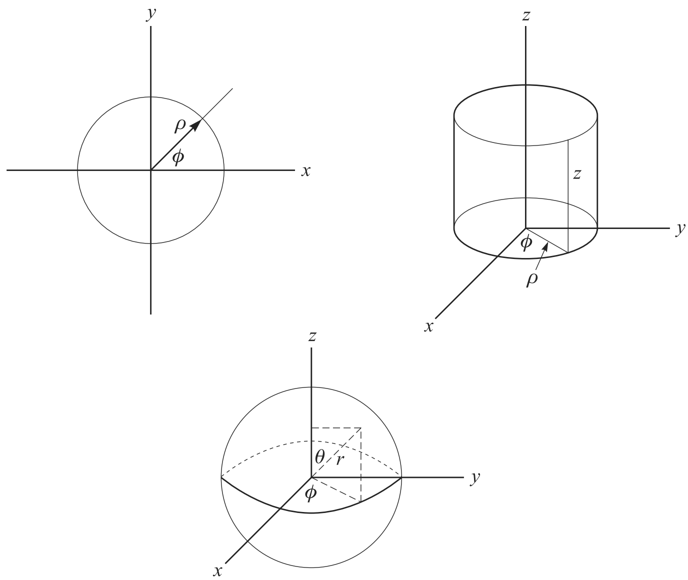

Fig. 9-5

In the cylindrical case $\mathbf{r} = \rho \cos \phi \mathbf{i} + \rho \sin \phi \mathbf{j} + z\mathbf{k}$ and the set is

$$\frac{\partial \mathbf{r}}{\partial \rho} = \cos \phi \mathbf{i} + \sin \phi \mathbf{j}, \quad \frac{\partial \mathbf{r}}{\partial \phi} = -\rho \sin \phi \mathbf{i} + \rho \cos \phi \mathbf{j}, \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}$$

Therefore $\frac{\partial \mathbf{r}}{\partial \rho} \cdot \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial z} = \rho.$

That the geometric interpretation of $\frac{\partial \mathbf{r}}{\partial \rho} \cdot \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial z}\, d\rho\, d\phi\, dz$ is an infinitesimal rectangular parallelepiped suggests the differential element of volume in cylindrical coordinates is

$$dV = \rho\, d\rho\, d\phi\, dz$$

Thus, for an integrable but otherwise arbitrary function, $F(\rho, \phi, z)$, of cylindrical coordinates, the iterated triple integral takes the form

$$\int_{z_1}^{z_2} \int_{g_1(z)}^{g_2(z)} \int_{f_1(\phi,z)}^{f_2(\phi,z)} F(\rho, \phi, z)\rho\, d\rho\, d\phi\, dz$$

The differential element of area for polar coordinates in the plane results by suppressing the z coordinate. It is

$$dA = \left| \frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \phi} \right| d\rho\, d\phi$$

and the iterated form of the double integral is

$$\int_{\rho_1}^{\rho_2} \int_{\phi_1(\rho)}^{\phi_2(\rho)} F(\rho, \phi)\rho\, d\rho\, d\phi$$

The transformation equations relating spherical and rectangular Cartesian coordinates are

$$x = r \sin \theta \cos \phi, \qquad y = r \sin \theta \sin \phi, \qquad z = r \cos \theta$$

In this case the coordinate surfaces are spheres, cones, and planes. (See Fig. 9-5.)

Following the same pattern as with cylindrical coordinates we discover that

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

and the iterated triple integral of $F(r, \theta, \phi)$ has the spherical representation

$$\int_{r_1}^{r_2} \int_{\theta_1(\phi)}^{\theta_2(\phi)} \int_{\phi_1(r,\theta)}^{\phi_2(r,\theta)} F(r, \theta, \phi) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Of course, the order of these integrations may be adapted to the geometry.

The coordinate surfaces in spherical coordinates are spheres, cones, and planes. If r is held constant, say, $r = a$, then we obtain the differential element of surface area

$$dA = a^2 \sin\theta \, d\theta \, d\phi$$

The first octant surface area of a sphere of radius a is

$$\int_0^{\pi/2} \int_0^{\pi/2} a^2 \sin\theta \, d\theta \, d\phi = \int_0^{\pi/2} a^2 (-\cos\theta)_0^{\frac{\pi}{2}} \, d\phi = \int_0^{\pi/2} a^2 \, d\phi = a^2 \frac{\pi}{2}$$

Thus, the surface area of the sphere is $4\pi a^2$.

Solved Problems

DOUBLE INTEGRALS

9.1. (*a*) Sketch the region $\mathscr{R}$ in the xy plane bounded by $y = x^2, x = 2, y = 1$.

(*b*) Give a physical interpreation to $\iint_{\mathscr{R}} (x^2 + y^2) \, dx \, dy$.

(*c*) Evaluate the double integral in (*b*).

(*a*) The required region $\mathscr{R}$ is shown shaded in Fig. 9-6 below.

(*b*) Since $x^2 + y^2$ is the square of the distance from any point (x, y) to $(0, 0)$, we can consider the double integral as representing the *polar moment of inertia* (i.e., moment of inertia with respect to the origin) of the region $\mathscr{R}$ (assuming unit density).

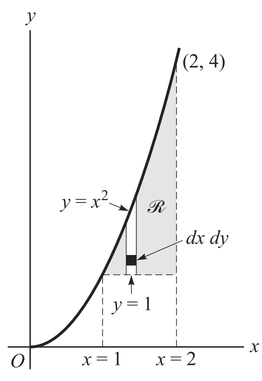

Fig. 9-6

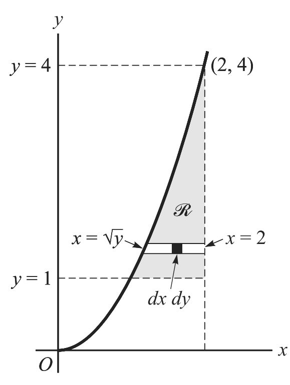

Fig. 9-7

We can also consider the double integral as representing the *mass* of the region $\mathscr{R}$ assuming a density varying as $x^2 + y^2$.

(*c*) **Method 1:** The double integral can be expressed as the iterated integral

$$\int_{x=1}^{2}\int_{y=1}^{x^2}(x^2+y^2)\,dy\,dx = \int_{x=1}^{2}\left\{\int_{y=1}^{x^2}(x^2+y^2)\,dy\right\}dx = \int_{x=1}^{2} x^2y+\frac{y^3}{3}\Big|_{y=1}^{x^2}\,dx$$
$$= \int_{x=1}^{2}\left(x^4+\frac{x^6}{3}-x^2-\frac{1}{3}\right)dx = \frac{1006}{105}$$

The integration with respect to y (keeping x constant) from $y = 1$ to $y = x^2$ corresponds formally to summing in a vertical column (see Fig. 9-6). The subsequent integration with respect to x from $x = 1$ to $x = 2$ corresponds to addition of contributions from all such vertical columns between $x = 1$ and $x = 2$.

Method 2: The double integral can also be expressed as the iterated integral

$$\int_{y=1}^{4}\int_{x=\sqrt{y}}^{2}(x^2+y^2)\,dx\,dy = \int_{y=1}^{4}\left\{\int_{x=\sqrt{y}}^{2}(x^2+y^2)\,dx\right\}dy = \int_{y=1}^{4}\frac{x^3}{3}+xy^2\Big|_{x=\sqrt{y}}^{2}\,dy$$
$$= \int_{y=1}^{4}\left(\frac{8}{3}+2y^2-\frac{y^{3/2}}{3}-y^{5/2}\right)dy = \frac{1006}{105}$$

In this case the vertical column of region $\mathscr{R}$ in Fig. 9-6 above is replaced by a horizontal column as in Fig. 9-7 above. Then the integration with respect to x (keeping y constant) from $x = \sqrt{y}$ to $x = 2$ corresponds to summing in this horizontal column. Subsequent integration with respect to y from $y = 1$ to $y = 4$ corresponds to addition of contributions for all such horizontal columns between $y = 1$ and $y = 4$.

9.2. Find the volume of the region bound by the elliptic paraboloid $z = 4 - x^2 - \frac{1}{4}y^2$ and the plane $z = 0$.

Because of the symmetry of the elliptic paraboloid, the result can be obtained by multiplying the first octant volume by 4.

Letting $z = 0$ yields $4x^2 + y^2 = 16$. The limits of integration are determined from this equation. The required volume is

$$4\int_0^2\int_0^{2\sqrt{4-x^2}}\left(4-x^2-\frac{1}{4}y^2\right)dy\,dx = 4\int_0^2\left(4y-x^2y-\frac{1}{4}\frac{y^3}{3}\right)_0^{2\sqrt{4-x^2}}dx$$
$$= 16\Pi$$

Hint: Use trigonometric substitutions to complete the integrations.

9.3. The geometric model of a material body is a plane region R bound by $y = x^2$ and $y = \sqrt{2 - x^2}$ on the interval $0 \leqq x \leqq 1$, and with a density function $\rho = xy$ (*a*) Draw the graph of the region. (*b*) Find the mass of the body. (*c*) Find the coordinates of the center of mass. (See Fig. 9-8.)

(*a*)

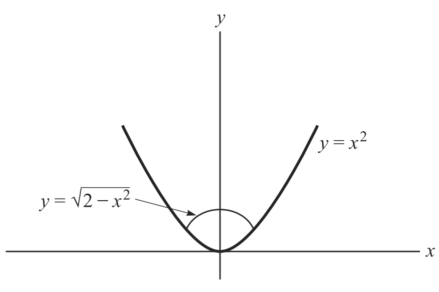

Fig. 9-8

(*b*)

$$M = \int_a^b \int_{f_1}^{f_2} \rho\, dy\, dx = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} yx\, dy\, dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{2-x^2}} x\, dx$$

$$= \int_0^1 \frac{1}{2} x(2 - x^2 - x^4)\, dx = \left[\frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{12}\right]_0^1 = \frac{7}{24}$$

(*c*) The coordinates of the center of mass are defined to be

$$\bar{x} = \frac{1}{M}\int_a^b \int_{f_1(x)}^{f_2(x)} x\, \rho\, dy\, dx \qquad \text{and} \qquad \bar{y} = \frac{1}{M}\int_a^b \int_{f_1(x)}^{f_2(x)} y\, \rho\, dy\, dx$$

where

$$M = \int_a^b \int_{f_1(x)}^{f_2(x)} \rho\, dy\, dx$$

Thus,

$$M\bar{x} = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} x\, xy\, dy\, dx = \int_0^1 x^2 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{2-x^2}} dx = \int_0^1 x^2 \frac{1}{2}[2 - x^2 - x^4]\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{10} - \frac{x^7}{14}\right]_0^1 = \frac{1}{3} - \frac{1}{10} - \frac{1}{14} = \frac{17}{105}$$

$$M\bar{y} = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} yx\, dy\, dx = -\frac{13}{120} + 4\frac{\sqrt{2}}{15}$$

9.4. Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

$$\begin{aligned} \text{Required volume} &= 8 \text{ times volume of region shown in Fig. 9-9} \\ &= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} z\, dy\, dx \\ &= 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy\, dx \\ &= 8 \int_{x=0}^{a} (a^2 - x^2)\, dx = \frac{16a^3}{3} \end{aligned}$$

As an aid in setting up this integral, note that $z\, dy\, dx$ corresponds to the volume of a column such as shown darkly shaded in the figure. Keeping x constant and integrating with respect to y from $y = 0$ to $y = \sqrt{a^2 - x^2}$ corresponds to adding the volumes of all such columns in a slab parallel to the yz plane, thus giving the volume of this slab. Finally, integrating with respect to x from $x = 0$ to $x = a$ corresponds to adding the volumes of all such slabs in the region, thus giving the required volume.

9.5. Find the volume of the region bounded by

$$z = x + y,\ z = 6,\ x = 0,\ y = 0,\ z = 0$$

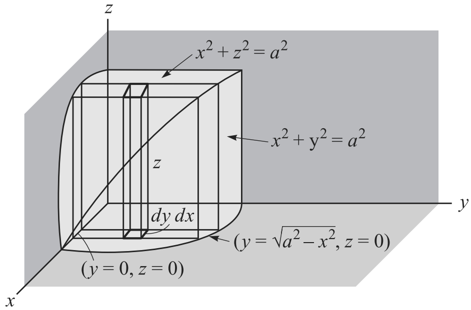

Fig. 9-9

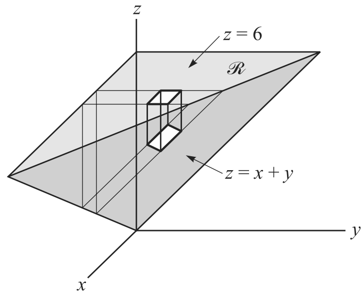

Fig. 9-10

$$\begin{aligned}\text{Required volume} &= \text{volume of region shown in Fig. 9-10}\\ &= \int_{x=0}^{6}\int_{y=0}^{6-x}\{6-(x+y)\}\,dy\,dx\\ &= \int_{x=0}^{6}(6-x)y-\frac{1}{2}y^2\Big|_{y=0}^{6-x}dx\\ &= \int_{x=0}^{6}\frac{1}{2}(6-x)^2\,dx = 36\end{aligned}$$

In this case the volume of a typical column (shown darkly shaded) corresponds to $\{6-(x+y)\}\,dy\,dx$. The limits of integration are then obtained by integrating over the region $\mathscr{R}$ of the figure. Keeping x constant and integrating with respect to y from $y=0$ to $y=6-x$ (obtained from $z=6$ and $z=x+y$) corresponds to summing all columns in a slab parallel to the yz plane. Finally, integrating with respect to x from $x=0$ to $x=6$ corresponds to adding the volumes of all such slabs and gives the required volume.

TRANSFORMATION OF DOUBLE INTEGRALS

9.6. Justify equation (*9*), Page 211, for changing variables in a double integral.

In rectangular coordinates, the double integral of $F(x,y)$ over the region $\mathscr{R}$ (shaded in Fig. 9-11) is $\iint_{\mathscr{R}} F(x,y)\,dx\,dy$. We can also evaluate this double integral by considering a grid formed by a family of u and v curvilinear coordinate curves constructed on the region $\mathscr{R}$ as shown in the figure.

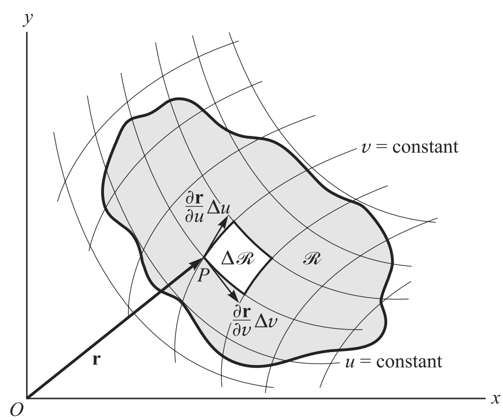

Fig. 9-11

Let P be any point with coordinates (x, y) or (u, v), where $x = f(u, v)$ and $y = g(u, v)$. Then the vector $\mathbf{r}$ from O to P is given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = f(u, v)\mathbf{i} + g(u, v)\mathbf{j}$. The tangent vectors to the coordinate curves $u = c_1$ and $v = c_2$, where c_1 and c_2 are constants, are $\partial\mathbf{r}/\partial v$ and $\partial\mathbf{r}/\partial u$, respectively. Then the area of region $\Delta\mathscr{R}$ of Fig. 9-11 is given approximately by $\left|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right| \Delta u\, \Delta v$.

But

$$\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k}$$

so that

$$\left|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right| \Delta u\, \Delta v = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\, \Delta v$$

The double integral is the limit of the sum

$$\sum F\{f(u, v), g(u, v)\}\left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\, \Delta v$$

taken over the entire region $\mathscr{R}$. An investigation reveals that this limit is

$$\iint_{\mathscr{R}'} F\{f(u, v), g(u, v)\}\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

where $\mathscr{R}'$ is the region in the uv plane into which the region $\mathscr{R}$ is mapped under the transformation $x = f(u, v)$, $y = g(u, v)$.

Another method of justifying the above method of change of variables makes use of line integrals and Green's theorem in the plane (see Chapter 10, Problem 10.32).

9.7. If $u = x^2 - y^2$ and $v = 2xy$, find $\partial(x, y)/\partial(u, v)$ in terms of u and v.

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

From the identify $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ we have

$$(x^2 + y^2)^2 = u^2 + v^2 \qquad \text{and} \qquad x^2 + y^2 = \sqrt{u^2 + v^2}$$

Then by Problem 6.43, Chapter 6,

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\partial(u, v)/\partial(x, y)} = \frac{1}{4(x^2 + y^2)} = \frac{1}{4\sqrt{u^2 + v^2}}$$

Another method: Solve the given equations for x and y in terms of u and v and find the Jacobian directly.

9.8. Find the polar moment of inertia of the region in the xy plane bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$, $xy = 4$ assuming unit density.

Under the transformation $x^2 - y^2 = u$, $2xy = v$ the required region $\mathscr{R}$ in the xy plane [shaded in Fig. 9-12(a)] is mapped into region $\mathscr{R}'$ of the uv plane [shaded in Fig. 9-12(b)]. Then:

$$\text{Required polar moment of inertia} = \iint_{\mathscr{R}} (x^2 + y^2)\, dx\, dy = \iint_{\mathscr{R}'} (x^2 + y^2)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

$$= \iint_{\mathscr{R}'} \sqrt{u^2 + v^2}\, \frac{du\, dv}{4\sqrt{u^2 + v^2}} = \frac{1}{4}\int_{u=1}^{9}\int_{v=4}^{8} du\, dv = 8$$

where we have used the results of Problem 9.7.

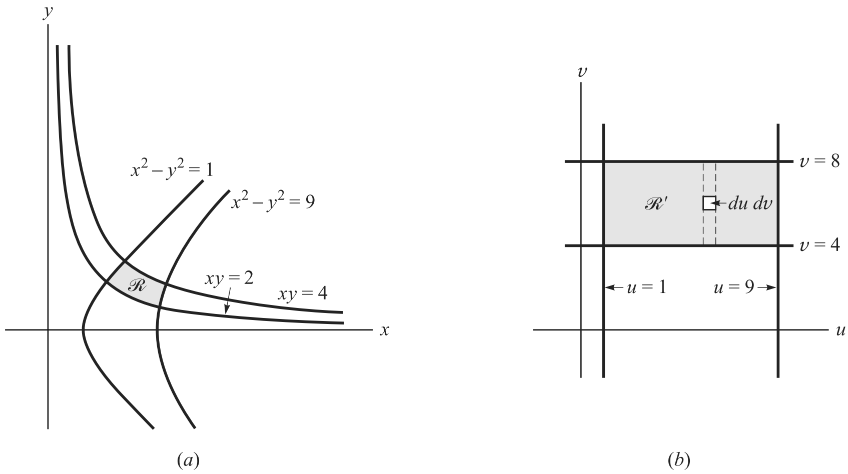

Fig. 9-12

Note that the limits of integration for the region $\mathscr{R}'$ can be constructed directly from the region $\mathscr{R}$ in the xy plane without actually constructing the region $\mathscr{R}'$. In such case we use a grid as in Problem 9.6. The coordinates (u, v) are curvilinear coordinates, in this case called *hyperbolic coordinates*.

9.9. Evaluate $\iint_{\mathscr{R}} \sqrt{x^2 + y^2}\, dx\, dy$, where $\mathscr{R}$ is the region in the xy plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

The presence of $x^2 + y^2$ suggests the use of polar coordinates (ρ, ϕ), where $x = \rho \cos \phi$, $y = \rho \sin \phi$ (see Problem 6.39, Chapter 6). Under this transformation the region $\mathscr{R}$ [Fig. 9-13(*a*) below] is mapped into the region $\mathscr{R}'$ [Fig. 9-13(*b*) below].

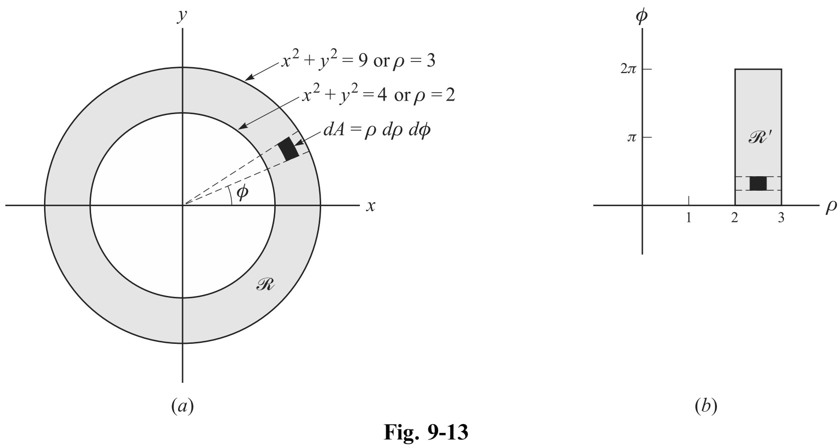

Fig. 9-13

Since $\dfrac{\partial(x, y)}{\partial(\rho, \phi)} = \rho$, it follows that

$$\iint_{\mathscr{R}} \sqrt{x^2 + y^2}\, dx\, dy = \iint_{\mathscr{R}'} \sqrt{x^2 + y^2} \left| \frac{\partial(x, y)}{\partial(\rho, \phi)} \right| d\rho\, d\phi = \iint_{\mathscr{R}'} \rho \cdot \rho\, d\rho\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\rho=2}^{3} \rho^2\, d\rho\, d\phi = \int_{\phi=0}^{2\pi} \frac{\rho^3}{3} \bigg|_2^3 d\phi = \int_{\phi=0}^{2\pi} \frac{19}{3}\, d\phi = \frac{38\pi}{3}$$

We can also write the integration limits for $\mathscr{R}'$ immediately on observing the region $\mathscr{R}$, since for fixed ϕ, ρ varies from $\rho = 2$ to $\rho = 3$ within the sector shown dashed in Fig. 9-13(*a*). An integration with respect to ϕ from $\phi = 0$ to $\phi = 2\pi$ then gives the contribution from all sectors. Geometrically, $\rho\, d\rho\, d\phi$ represents the area dA as shown in Fig. 9-13(*a*).

9.10. Find the area of the region in the xy plane bounded by the lemniscate $\rho^2 = a^2 \cos 2\phi$.

Here the curve is given directly in polar coordinates (ρ, ϕ). By assigning various values to ϕ and finding corresponding values of ρ, we obtain the graph shown in Fig. 9-14. The required area (making use of symmetry) is

$$4\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{a\sqrt{\cos 2\phi}} \rho\, d\rho\, d\phi = 4\int_{\phi=0}^{\pi/4} \left.\frac{\rho^3}{2}\right|_{\rho=0}^{a\sqrt{\cos 2\phi}} d\phi$$

$$= 2\int_{\phi=0}^{\pi/4} a^2 \cos 2\phi\, d\phi = a^2 \sin 2\phi\Big|_{\phi=0}^{\pi/4} = a^2$$

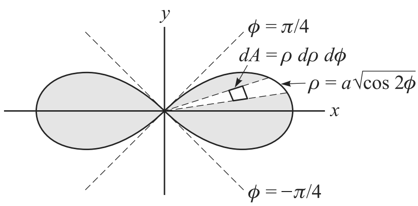

Fig. 9-14

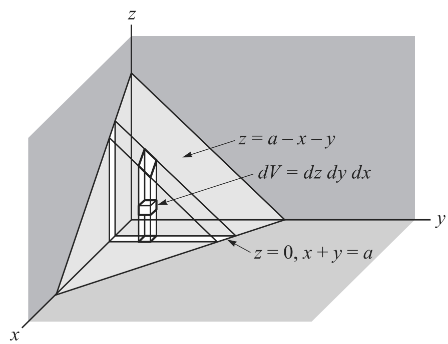

Fig. 9-15

TRIPLE INTEGRALS

9.11. (*a*) Sketch the three-dimensional region $\mathscr{R}$ bounded by $x + y + z = a$ $(a > 0)$, $x = 0$, $y = 0$, $z = 0$.
(*b*) Give a physical interpretation to

$$\iiint_{\mathscr{R}} (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

(*c*) Evaluate the triple integral in (*b*).

(*a*) The required region $\mathscr{R}$ is shown in Fig. 9-15.

(*b*) Since $x^2 + y^2 + z^2$ is the square of the distance from any point (x, y, z) to $(0, 0, 0)$, we can consider the triple integral as representing the *polar moment of inertia* (i.e., moment of inertia with respect to the origin) of the region $\mathscr{R}$ (assuming unit density).

We can also consider the triple integral as representing the *mass* of the region if the density varies as $x^2 + y^2 + z^2$.

(*c*) The triple integral can be expressed as the iterated integral

$$\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x^2+y^2+z^2)\,dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} x^2z+y^2z+\frac{z^3}{3}\bigg|_{z=0}^{a-x-y} dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left\{x^2(a-x)-x^2y+(a-x)y^2-y^3+\frac{(a-x-y)^3}{3}\right\}dy\,dx$$

$$= \int_{x=0}^{a} x^2(a-x)y-\frac{x^2y^2}{2}+\frac{(a-x)y^3}{3}-\frac{y^4}{4}-\frac{(a-x-y)^4}{12}\bigg|_{y=0}^{a-x} dx$$

$$= \int_0^a\left\{x^2(a-x)^2-\frac{x^2(a-x)^2}{2}+\frac{(a-x)^4}{3}-\frac{(a-x)^4}{4}+\frac{(a-x)^4}{12}\right\}dx$$

$$= \int_0^a\left\{\frac{x^2(a-x)^2}{2}+\frac{(a-x)^4}{6}\right\}dx=\frac{a^5}{20}$$

The integration with respect to z (keeping x and y constant) from $z=0$ to $z=a-x-y$ corresponds to summing the polar moments of inertia (or masses) corresponding to each cube in a vertical column. The subsequent integration with respect to y from $y=0$ to $y=a-x$ (keeping x constant) corresponds to addition of contributions from all vertical columns contained in a slab parallel to the yz plane. Finally, integration with respect to x from $x=0$ to $x=a$ adds up contributions from all slabs parallel to the yz plane.

Although the above integration has been accomplished in the order z, y, x, any other order is clearly possible and the final answer should be the same.

9.12. Find the (*a*) volume and (*b*) centroid of the region $\mathscr{R}$ bounded by the parabolic cylinder $z=4-x^2$ and the planes $x=0$, $y=0$, $y=6$, $z=0$ assuming the density to be a constant σ.

The region $\mathscr{R}$ is shown in Fig. 9-16.

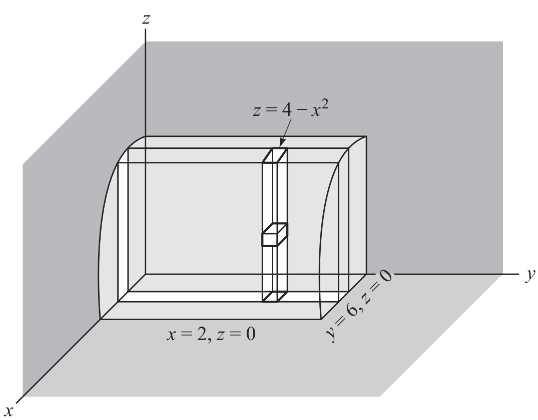

Fig. 9-16

(*a*) Required volume $= \iiint_{\mathscr{R}} dx\,dy\,dz$

$$= \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} dz\,dy\,dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{6} (4 - x^2)\,dy\,dx$$

$$= \int_{x=0}^{2} (4 - x^2)y\Big|_{y=0}^{6}\,dx$$

$$= \int_{x=0}^{2} (24 - 6x^2)\,dx = 32$$

(*b*) Total mass $= \int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} \sigma\,dz\,dy\,dx = 32\sigma$ by part (*a*), since σ is constant. Then

$$\bar{x} = \frac{\text{Total moment about } yz \text{ plane}}{\text{Total mass}} = \frac{\int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} \sigma x\,dz\,dy\,dx}{\text{Total mass}} = \frac{24\sigma}{32\sigma} = \frac{3}{4}$$

$$\bar{y} = \frac{\text{Total moment about } xz \text{ plane}}{\text{Total mass}} = \frac{\int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} \sigma y\,dz\,dy\,dx}{\text{Total mass}} = \frac{96\sigma}{32\sigma} = 3$$

$$\bar{z} = \frac{\text{Total moment about } xy \text{ plane}}{\text{Total mass}} = \frac{\int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} \sigma z\,dz\,dy\,dx}{\text{Total mass}} = \frac{256\sigma/5}{32\sigma} = \frac{8}{5}$$

Thus, the centroid has coordinates $(3/4, 3, 8/5)$.

Note that the value for $\bar{y}$ could have been predicted because of symmetry.

TRANSFORMATION OF TRIPLE INTEGRALS

9.13. Justify equation (*11*), Page 211, for changing variables in a triple integral.

By analogy with Problem 9.6, we construct a grid of curvilinear coordinate surfaces which subdivide the region $\mathscr{R}$ into subregions, a typical one of which is $\Delta\mathscr{R}$ (see Fig. 9-17).

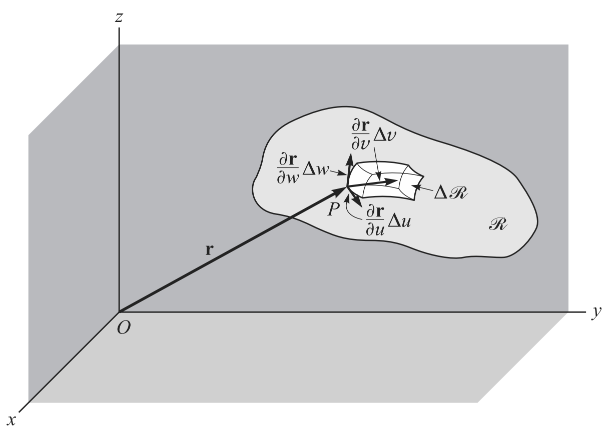

Fig. 9-17

The vector **r** from the origin O to point P is

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(u, v, w)\mathbf{i} + g(u, v, w)\mathbf{j} + h(u, v, w)\mathbf{k}$$

assuming that the transformation equations are $x = f(u, v, w)$, $y = g(u, v, w)$, and $z = h(u, v, w)$.

Tangent vectors to the coordinate curves corresponding to the intersection of pairs of coordinate surfaces are given by $\partial\mathbf{r}/\partial u$, $\partial\mathbf{r}/\partial v$, $\partial\mathbf{r}/\partial w$. Then the volume of the region $\Delta\mathscr{R}$ of Fig. 9-17 is given approximately by

$$\left|\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right|\Delta u\,\Delta v\,\Delta w = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|\Delta u\,\Delta v\,\Delta w$$

The triple integral of $F(x, y, z)$ over the region is the limit of the sum

$$\sum F\{f(u, v, w), g(u, v, w), h(u, v, w)\}\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right|\Delta u\,\Delta v\,\Delta w$$

An investigation reveals that this limit is

$$\iiint_{\mathscr{R}'} F\{f(u, v, w), g(u, v, w), h(u, v, w)\}\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw$$

where $\mathscr{R}'$ is the region in the uvw space into which the region $\mathscr{R}$ is mapped under the transformation.

Another method for justifying the above change of variables in triple integrals makes use of Stokes' theorem (see Problem 10.84, Chapter 10).

9.14. What is the mass of a circular cylindrical body represented by the region $0 \leqq \rho \leqq c, 0 \leqq \phi \leqq 2\pi, 0 \leqq z \leqq h$, and with the density function $\mu = z\sin^2\phi$?

$$M = \int_0^h \int_0^{2\pi} \int_0^c z\sin^2\phi\rho\,d\rho\,d\phi\,dz = \pi$$

9.15. Use spherical coordinates to calculate the volume of a sphere of radius a.

$$V = 8\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} a^2\sin\theta\,dr\,d\theta\,d\phi = \frac{4}{3}\pi a^3$$

9.16. Express $\iiint_{\mathscr{R}} F(x, y, z)\,dx\,dy\,dz$ in (*a*) cylindrical and (*b*) spherical coordinates.

(*a*) The transformation equations in cylindrical coordinates are $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

As in Problem 6.39, Chapter 6, $\partial(x, y, z)/\partial(\rho, \phi, z) = \rho$. Then by Problem 9.13 the triple integral becomes

$$\iiint_{\mathscr{R}'} G(\rho, \phi, z)\,\rho\,d\rho\,d\phi\,dz$$

where $\mathscr{R}'$ is the region in the ρ, ϕ, z space corresponding to $\mathscr{R}$ and where $G(\rho, \phi, z \equiv F(\rho\cos\phi, \rho\sin\phi, z)$.

(*b*) The transformation equations in spherical coordinates are $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

By Problem 6.101, Chapter 6, $\partial(x, y, z)/\partial(r, \theta, \phi) = r^2\sin\theta$. Then by Problem 9.13 the triple integral becomes

$$\iiint_{\mathscr{R}'} H(r, \theta, \phi) r^2 \sin\theta\,dr\,d\theta\,d\phi$$

where $\mathscr{R}'$ is the region in the r, θ, ϕ space corresponding to $\mathscr{R}$, and where $H(r, \theta, \phi) \equiv F(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$.

9.17. Find the volume of the region above the xy plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$.

The volume is most easily found by using cylindrical coordinates. In these coordinates the equations for the paraboloid and cylinder are respectively $z = \rho^2$ and $\rho = a$. Then

$$\begin{aligned}
\text{Required volume} &= 4 \text{ times volume shown in Fig. 9-18} \\
&= 4 \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} \int_{z=0}^{\rho^2} \rho \, dz \, d\rho \, d\phi \\
&= 4 \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} \rho^3 \, d\rho \, d\phi \\
&= 4 \int_{hi=0}^{\pi/2} \left. \frac{\rho^4}{4} \right|_{=0}^{a} d\phi = \frac{\pi}{2} a^4
\end{aligned}$$

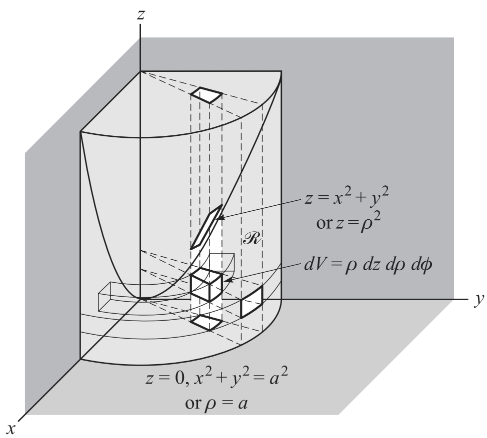

Fig. 9-18

The integration with respect to z (keeping ρ and ϕ constant) from $z = 0$ to $z = \rho^2$ corresponds to summing the cubical volumes (indicated by dV) in a vertical column extending from the xy plane to the paraboloid. The subsequent integration with respect to ρ (keeping ϕ constant) from $\rho = 0$ to $\rho = a$ corresponds to addition of volumes of all columns in the wedge-shaped region. Finally, integration with respect to ϕ corresponds to adding volumes of all such wedge-shaped regions.

The integration can also be performed in other orders to yield the same result.

We can also set up the integral by determining the region $\mathscr{R}'$ in ρ, ϕ, z space into which $\mathscr{R}$ is mapped by the cylindrical coordinate transformation.

9.18. (*a*) Find the moment of inertia about the z-axis of the region in Problem 9.17, assuming that the density is the constant σ. (*b*) Find the radius of gyration.

(*a*) The moment of inertia about the z-axis is

$$\begin{aligned}
I_z &= 4 \int_{\phi_0}^{\pi/2} \int_{\rho=0}^{a} \int_{z=0}^{\rho^2} \rho^2 \cdot \sigma \rho \, dz \, d\rho \, d\phi \\
&= 4\sigma \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} \rho^5 \, d\rho \, d\phi = 4\sigma \int_{\phi=0}^{\pi/2} \left. \frac{\rho^6}{6} \right|_{\rho=0}^{a} d\phi = \frac{\pi a^6 \sigma}{3}
\end{aligned}$$

The result can be expressed in terms of the mass M of the region, since by Problem 9.17,

$$M = \text{volume} \times \text{density} = \frac{\pi}{2}a^4\sigma \quad \text{so that} \quad I_z = \frac{\pi a^6 \sigma}{3} = \frac{\pi a^6}{3} \cdot \frac{2M}{\pi a^4} = \frac{2}{3}Ma^2$$

Note that in setting up the integral for I_z we can think of $\sigma\rho\, dz\, d\rho\, d\phi$ as being the mass of the cubical volume element, $\rho^2 \cdot \sigma\rho\, dz\, d\rho\, d\phi$, as the moment of inertia of this mass with respect to the z-axis and $\iiint_{\mathscr{R}} \rho^2 \cdot \sigma\rho\, dz\, d\rho\, d\phi$ as the total moment of inertia about the z-axis. The limits of integration are determined as in Problem 9.17.

(*b*) The radius of gyration is the value K such that $MK^2 = \frac{2}{3}Ma^2$, i.e., $K^2 = \frac{2}{3}a^2$ or $K = a\sqrt{2/3}$.

The physical significance of K is that if all the mass M were concentrated in a thin cylindrical shell of radius K, then the moment of inertia of this shell about the axis of the cylinder would be I_z.

9.19. (*a*) Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $z^2 \sin^2 \alpha = (x^2 + y^2)\cos^2 \alpha$, where α is a constant such that $0 \leqq \alpha \leqq \pi$. (*b*) From the result in (*a*), find the volume of a sphere of radius a.

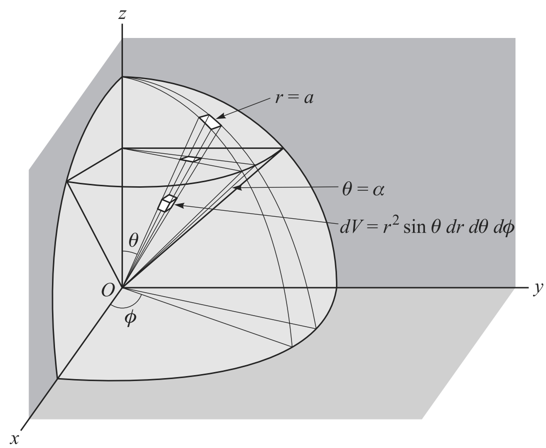

Fig. 9-19

In spherical coordinates the equation of the sphere is $r = a$ and that of the cone is $\theta = \alpha$. This can be seen directly or by using the transformation equations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. For example, $z^2 \sin^2 \alpha = (x^2 + y^2)\cos^2 \alpha$ becomes, on using these equations,

$$r^2\cos^2\theta\sin^2\alpha = (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi)\cos^2\alpha$$

i.e., $r^2\cos^2\theta\sin^2\alpha = r^2\sin^2\theta\cos^2\alpha$

from which $\tan\theta = \pm\tan\alpha$ and so $\theta = \alpha$ or $\theta = \pi - \alpha$. It is sufficient to consider one of these, say, $\theta = \alpha$.

(*a*) Required volume = 4 times volume (shaded) in Fig. 9-19

$$\begin{aligned}
&= 4\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha}\int_{r=0}^{a} r^2\sin\theta\, dr\, d\theta\, d\phi \\
&= 4\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha} \frac{r^3}{3}\sin\theta\Big|_{r=0}^{\alpha} d\theta\, d\phi \\
&= \frac{4a^3}{3}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha} \sin\theta\, d\theta\, d\phi \\
&= \frac{4a^3}{3}\int_{\phi=0}^{\pi/2} -\cos\theta\Big|_{\theta=0}^{\alpha} d\phi \\
&= \frac{2\pi a^3}{3}(1 - \cos\alpha)
\end{aligned}$$

The integration with respect to r (keeping θ and ϕ constant) from $r = 0$ to $r = a$ corresponds to summing the volumes of all cubical elements (such as indicated by dV) in a column extending from $r = 0$ to $r = a$. The subsequent integration with respect to θ (keeping ϕ constant) from $\theta = 0$ to $\theta = \pi/4$ corresponds to summing the volumes of all columns in the wedge-shaped region. Finally, integration with respect to ϕ corresponds to adding volumes of all such wedge-shaped regions.

(*b*) Letting $\alpha = \pi$, the volume of the sphere thus obtained is

$$\frac{2\pi a^3}{3}(1 - \cos\pi) = \frac{4}{3}\pi a^3$$

9.20. (*a*) Find the centroid of the region in Problem 9.19.
(*b*) Use the result in (*a*) to find the centroid of a hemisphere.

(*a*) The centroid $(\bar{x}, \bar{y}, \bar{z})$ is, due to symmetry, given by $\bar{x} = \bar{y} = 0$ and

$$\bar{z} = \frac{\text{Total moment about } xy \text{ plane}}{\text{Total mass}} = \frac{\iiint z\sigma\, dV}{\iiint \sigma\, dV}$$

Since $z = r\cos\theta$ and σ is constant the numerator is

$$\begin{aligned}
4\sigma \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \int_{r=0}^{a} r\cos\theta \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi &= 4\sigma \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \frac{r^4}{4}\bigg|_{r=0}^{a} \sin\theta\cos\theta\, d\theta\, d\phi \\
&= \sigma a^4 \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\alpha} \sin\theta\cos\theta\, d\theta\, d\phi \\
&= \sigma a^4 \int_{\phi=0}^{\pi/2} \frac{\sin^2\theta}{2}\bigg|_{\theta=0}^{\alpha} d\phi = \frac{\pi\sigma a^4 \sin^2\alpha}{4}
\end{aligned}$$

The denominator, obtained by multiplying the result of Problem 9.19(*a*) by σ, is $\frac{2}{3}\pi\sigma a^3(1 - \cos\alpha)$. Then

$$\bar{z} = \frac{\frac{1}{4}\pi\sigma a^4 \sin^2\alpha}{\frac{2}{3}\pi\sigma a^3(1 - \cos\alpha)} = \frac{3}{8}a(1 + \cos\alpha).$$

(*b*) Letting $\alpha = \pi/2$, $\bar{z} = \frac{3}{8}a$.

MISCELLANEOUS PROBLEMS

9.21. Prove that (*a*) $\displaystyle\int_0^1 \left\{\int_0^1 \frac{x-y}{(x+y)^3}\, dy\right\} dx = \frac{1}{2}$, (*b*) $\displaystyle\int_0^1 \left\{\int_0^1 \frac{x-y}{(x+y)^3}\, dx\right\} dy = -\frac{1}{2}$.

(*a*)
$$\begin{aligned}
\int_0^1 \left\{\int_0^1 \frac{x-y}{(x+y)^3}\, dy\right\} dx &= \int_0^1 \left\{\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\, dy\right\} dx \\
&= \int_0^1 \left\{\int_0^1 \left(\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right) dy\right\} dx \\
&= \int_0^1 \left(\frac{-x}{(x+y)^2} + \frac{1}{x+y}\right)\bigg|_{y=0}^{1} dx \\
&= \int_0^1 \frac{dx}{(x+1)^2} = \frac{-1}{x+1}\bigg|_0^1 = \frac{1}{2}
\end{aligned}$$

(*b*) This follows at once on formally interchanging x and y in (*a*) to obtain $\displaystyle\int_0^1 \left\{\int_0^1 \frac{y-x}{(x+y)^3}\, dx\right\} dy = \frac{1}{2}$ and then multiplying both sides by -1.

This example shows that interchange in order of integration may not always produce equal results. A sufficient condition under which the order may be interchanged is that the double integral over the corresponding region exists. In this case $\displaystyle\iint_{\mathscr{R}} \frac{x-y}{(x+y)^3}\, dx\, dy$, where $\mathscr{R}$ is the region $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$ fails to exist because of the discontinuity of the integrand at the origin. The integral is actually an *improper* double integral (see Chapter 12).

9.22. Prove that $\displaystyle\int_0^x \left\{\int_0^t F(u)\, du\right\} dt = \int_0^x (x-u)F(u)\, du.$

Let $I(x) = \int_0^x \left\{ \int_0^t F(u)\,du \right\} dt, \quad J(x) = \int_0^z (x-u)F(u)\,du.$ Then

$$I'(x) = \int_0^z F(u)\,du, \quad J'(x) = \int_0^z F(u)\,du$$

using Leibnitz's rule, Page 186. Thus, $I'(x) = J'(x)$, and so $I(x) - J(x) = c$, where c is a constant. Since $I(0) = J(0) = 0$, $c = 0$, and so $I(x) = J(x)$.

The result is sometimes written in the form

$$\int_0^x \int_0^x F(x)\,dx^2 = \int_0^x (x-u)F(u)\,du$$

The result can be generalized to give (see Problem 9.58)

$$\int_0^x \int_0^x \cdots \int_0^x F(x)\,dx^n = \frac{1}{(n-1)!} \int_0^x (x-u)^{n-1} F(u)\,du$$

Supplementary Problems

DOUBLE INTEGRALS

9.23. (*a*) Sketch the region $\mathscr{R}$ in the xy plane bounded by $y^2 = 2x$ and $y = x$. (*b*) Find the area of $\mathscr{R}$. (*c*) Find the polar moment of inertia of $\mathscr{R}$ assuming constant density σ.
Ans. (*b*) $\frac{2}{3}$; (*c*) $48\sigma/35 = 72M/35$, where M is the mass of $\mathscr{R}$.

9.24. Find the centroid of the region in the preceding problem. *Ans.* $\bar{x} = \frac{4}{5}, \bar{y} = 1$

9.25. Given $\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\,dx\,dy$. (*a*) Sketch the region and give a possible physical interpretation of the double integral. (*b*) Interchange the order of integration. (*c*) Evaluate the double integral.

Ans. (*b*) $\int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\,dy\,dx$, (*c*) $241/60$

9.26. Show that $\int_{x=1}^{2} \int_{y=\sqrt{x}}^{x} \sin \frac{\pi x}{2y}\,dy\,dx + \int_{x=2}^{4} \int_{y=\sqrt{x}}^{2} \sin \frac{\pi x}{2y}\,dy\,dx = \frac{4(\pi+2)}{\pi^3}$.

9.27. Find the volume of the tetrahedron bounded by $x/a + y/b + z/c = 1$ and the coordinate planes.
Ans. $abc/6$

9.28. Find the volume of the region bounded by $z = x^3 + y^2, z = 0, x = -a, x = a, y = -a, y = a$.
Ans. $8a^4/3$

9.29. Find (*a*) the moment of inertia about the z-axis and (*b*) the centroid of the region in Problem 9.28 assuming a constant density σ.
Ans. (*a*) $\frac{112}{45}a^6\sigma = \frac{14}{15}Ma^2$, where M = mass; (*b*) $\bar{x} = \bar{y} = 0, \bar{z} = \frac{7}{15}a^2$

TRANSFORMATION OF DOUBLE INTEGRALS

9.30. Evaluate $\iint_{\mathscr{R}} \sqrt{x^2+y^2}\,dx\,dy$, where $\mathscr{R}$ is the region $x^2 + y^2 \leqq a^2$. *Ans.* $\frac{2}{3}\pi a^3$

9.31. If $\mathscr{R}$ is the region of Problem 9.30, evaluate $\iint_{\mathscr{R}} e^{-(x^2+y^2)}\,dx\,dy$. *Ans.* $\pi(1-e^{-a^2})$

9.32. By using the transformation $x+y=u$, $y=uv$, show that

$$\int_{x=0}^{1}\int_{y=0}^{1-x} e^{y/(x+y)}\,dy\,dx = \frac{e-1}{2}$$

9.33. Find the area of the region bounded by $xy=4$, $xy=8$, $xy^3=5$, $xy^3=15$. [Hint: Let $xy=u$, $xy^3=v$.] *Ans.* $2\ln 3$

9.34. Show that the volume generated by revolving the region in the first quadrant bounded by the parabolas $y^2=x$, $y^2=8x$, $x^2=y$, $x^2=8y$ about the x-axis is $279\pi/2$. [Hint: Let $y^2=ux$, $x^2=vy$.]

9.35. Find the area of the region in the first quadrant bounded by $y=x^3$, $y=4x^3$, $x=y^3$, $x=4y^3$. *Ans.* $\frac{1}{8}$

9.36. Let $\mathscr{R}$ be the region bounded by $x+y=1$, $x=0$, $y=0$. Show that $\iint_{\mathscr{R}} \cos\left(\frac{x-y}{x+y}\right)dx\,dy = \frac{\sin 1}{2}$. [Hint: Let $x-y=u$, $x+y=v$.]

TRIPLE INTEGRALS

9.37. (*a*) Evaluate $\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} xyz\,dz\,dy\,dx$. (*b*) Give a physical interpretation to the integral in (*a*).

Ans. (*a*) $\frac{3}{8}$

9.38. Find the (*a*) volume and (*b*) centroid of the region in the first octant bounded by $x/a+y/b+z/c=1$, where a, b, c are positive. *Ans.* (*a*) $abc/6$; (*b*) $\bar{x}=a/4$, $\bar{y}=b/4$, $\bar{z}=c/4$

9.39. Find the (*a*) moment of inertia and (*b*) radius of gyration about the z-axis of the region in Problem 9.38. *Ans.* (*a*) $M(a^2+b^2)/10$, (*b*) $\sqrt{(a^2+b^2)/10}$

9.40. Find the mass of the region corresponding to $x^2+y^2+z^2 \leqq 4$, $x \geqq 0$, $y \geqq 0$, $z \geqq 0$, if the density is equal to xyz. *Ans.* $4/3$

9.41. Find the volume of the region bounded by $z=x^2+y^2$ and $z=2x$. *Ans.* $\pi/2$

TRANSFORMATION OF TRIPLE INTEGRALS

9.42. Find the volume of the region bounded by $z=4-x^2-y^2$ and the xy plane. *Ans.* 8π

9.43. Find the centroid of the region in Problem 9.42, assuming constant density σ. *Ans.* $\bar{x}=\bar{y}=0$, $\bar{z}=\frac{4}{3}$

9.44. (*a*) Evaluate $\iiint_{\mathscr{R}} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, where $\mathscr{R}$ is the region bounded by the plane $z=3$ and the cone $z=\sqrt{x^2+y^2}$. (*b*) Give a physical interpretation of the integral in (*a*). [Hint: Perform the integration in cylindrical coordinates in the order ρ, z, ϕ.] *Ans.* $27\pi(2\sqrt{2}-1)/2$

9.45. Show that the volume of the region bonded by the cone $z=\sqrt{x^2+y^2}$ and the paraboloid $z=x^2+y^2$ is $\pi/6$.

9.46. Find the moment of inertia of a right circular cylinder of radius a and height b, about its axis if the density is proportional to the distance from the axis. *Ans.* $\frac{3}{5}Ma^2$

9.47. (*a*) Evaluate $\iiint_{\mathscr{R}} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, where $\mathscr{R}$ is the region bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ where $a>b>0$. (*b*) Give a physical interpretation of the integral in (*a*).
Ans. (*a*) $4\pi\ln(a/b)$

9.48. (*a*) Find the volume of the region bounded above by the sphere $r=2a\cos\theta$, and below by the cone $\phi=\alpha$ where $0<\alpha<\pi/2$. (*b*) Discuss the case $\alpha=\pi/2$. *Ans.* $\frac{4}{3}\pi a^3(1-\cos^4\alpha)$

9.49. Find the centroid of a hemispherical shell having outer radius a and inner radius b if the density (*a*) is constant, (*b*) varies as the square of the distance from the base. Discuss the case $a=b$.
Ans. Taking the z-axis as axis of symmetry: (*a*) $\bar{x}=\bar{y}=0, \bar{z}=\frac{3}{8}(a^4-b^4)/(a^3-b^3)$; (*b*) $\bar{x}=\bar{y}=0$, $\bar{z}=\frac{5}{8}(a^6-b^6)/(a^5-b^5)$

MISCELLANEOUS PROBLEMS

9.50. Find the mass of a right circular cylinder of radius a and height b if the density varies as the square of the distance from a point on the circumference of the base.
Ans. $\frac{1}{6}\pi a^2bk(9a^2+2b^2)$, where k = constant of proportionality.

9.51. Find the (*a*) volume and (*b*) centroid of the region bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the plane $z=b$ where $a>b>0$, assuming constant density.
Ans. (*a*) $\frac{1}{3}\pi(2a^3-3a^2b+b^3)$; (*b*) $\bar{x}=\bar{y}=0, \bar{z}=\frac{3}{4}(a+b)^2/(2a+b)$

9.52. A sphere of radius a has a cylindrical hole of radius b bored from it, the axis of the cylinder coinciding with a diameter of the sphere. Show that the volume of the sphere which remains is $\frac{4}{3}\pi[a^3-(a^2-b^2)^{3/2}]$.

9.53. A simple closed curve in a plane is revolved about an axis in the plane which does not intersect the curve. Prove that the volume generated is equal to the area bounded by the curve multiplied by the distance traveled by the centroid of the area (*Pappus' theorem*).

9.54. Use Problem 9.53 to find the volume generated by revolving the circle $x^2+(y-b)^2=a^2, b>a>0$ about the x-axis. *Ans.* $2\pi^2a^2b$

9.55. Find the volume of the region bounded by the hyperbolic cylinders $xy=1, xy=9, xz=4, xz=36, yz=25$, $yz=49$. [Hint: Let $xy=u, xz=v, yz=w$.] *Ans.* 64

9.56. Evaluate $\iiint_{\mathscr{R}} \sqrt{1-(x^2/a^2+y^2/b^2+z^2/c^2)}\,dx\,dy\,dz$, where $\mathscr{R}$ is the region interior to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. [Hint: Let $x=au, y=bv, z=cw$. Then use spherical coordinates.]
Ans. $\frac{1}{4}\pi^2abc$

9.57. If $\mathscr{R}$ is the region $x^2+xy+y^2\leqq 1$, prove that $\iint_{\mathscr{R}} e^{-(x^2+xy+y^2)}\,dx\,dy=\frac{2\pi}{e\sqrt{3}}(e-1)$. [Hint: Let $x=u\cos\alpha-v\sin\alpha$, $y=u\sin\alpha+v\cos\alpha$ and choose α so as to eliminate the xy term in the integrand. Then let $u=a\rho\cos\phi$, $v=b\rho\sin\phi$ where a and b are appropriately chosen.]

9.58. Prove that $\int_0^x\int_0^x\cdots\int_0^x F(x)\,dx^n=\frac{1}{(n-1)!}\int_0^x(x-u)^{n-1}F(u)\,du$ for $n=1,2,3,\ldots$ (see Problem 9.22).